



TITLE:

Sufficient conditions for the global rigidity of graphs

AUTHOR(S):

Tanigawa, Shin-ichi

CITATION:

Tanigawa, Shin-ichi. Sufficient conditions for the global rigidity of graphs. Journal of Combinatorial Theory, Series B 2015, 113: 123-140

ISSUE DATE:

2015-07

URL:

<http://hdl.handle.net/2433/201408>

RIGHT:

©2015. This manuscript version is made available under the CC-BY-NC-ND 4.0 license <http://creativecommons.org/licenses/by-nc-nd/4.0/>; The full-text file will be made open to the public on 31 July 2017 in accordance with publisher's 'Terms and Conditions for Self-Archiving'; This is not the published version. Please cite only the published version.; この論文は出版社版ではありません。引用の際には出版社版をご確認ご利用ください。

Sufficient Conditions for the Global Rigidity of Graphs

Shin-ichi Tanigawa*

January 10, 2015

Abstract

We investigate how to find generic and globally rigid realizations of graphs in \mathbb{R}^d based on elementary geometric observations. Our arguments lead to new proofs of a combinatorial characterization of the global rigidity of graphs in \mathbb{R}^2 by Jackson and Jordán and that of body-bar graphs in \mathbb{R}^d recently shown by Connelly, Jordán, and Whiteley. We also extend the 1-extension theorem and Connelly's composition theorem, which are main tools for generating globally rigid graphs in \mathbb{R}^d . In particular we show that any vertex-redundantly rigid graph in \mathbb{R}^d is globally rigid in \mathbb{R}^d , where a graph $G = (V, E)$ is called vertex-redundantly rigid if $G - v$ is rigid for any $v \in V$.

Keywords: rigidity of graphs, global rigidity, unique graph realizations, rigidity matroid

1 Introduction

A d -dimensional *bar-joint framework* (or simply a framework) is a pair (G, \mathbf{p}) of a graph $G = (V, E)$ and a map $\mathbf{p} : v \in V \mapsto p_v \in \mathbb{R}^d$. A pair (G, \mathbf{p}) is also called a *realization* of G in \mathbb{R}^d . We say that (G, \mathbf{p}) is *equivalent* to (G, \mathbf{q}) , denoted $(G, \mathbf{p}) \simeq (G, \mathbf{q})$, if

$$\|p_u - p_v\| = \|q_u - q_v\| \quad \text{for all } uv \in E, \quad (1)$$

and we say that \mathbf{p} and \mathbf{q} are *congruent*, denoted $\mathbf{p} \equiv \mathbf{q}$, if

$$\|p_u - p_v\| = \|q_u - q_v\| \quad \text{for all pairs } u, v \in V.$$

The set of all maps \mathbf{p} of the form $\mathbf{p} : V \rightarrow \mathbb{R}^d$ is denoted by \mathbb{R}^{dV} , and we will regard \mathbb{R}^{dV} as the $d|V|$ -dimensional Euclidean space. A framework (G, \mathbf{p}) is called *rigid* if there is an open neighborhood U of \mathbf{p} in \mathbb{R}^{dV} such that $\mathbf{p} \equiv \mathbf{q}$ holds for any $\mathbf{q} \in U$ with $(G, \mathbf{p}) \simeq (G, \mathbf{q})$. (G, \mathbf{p}) is called *globally rigid* if $\mathbf{p} \equiv \mathbf{q}$ holds for any $\mathbf{q} \in \mathbb{R}^{dV}$ with $(G, \mathbf{p}) \simeq (G, \mathbf{q})$.

A map $\mathbf{p} \in \mathbb{R}^{dV}$ is called *generic* if the set of entries in \mathbf{p} is algebraically independent over \mathbb{Q} . A framework (G, \mathbf{p}) is called *generic* if \mathbf{p} is generic. As observed by Gluck [12] and Asimov and Roth [1], a generic realization of G is rigid in \mathbb{R}^d if and only if every generic realization of G is rigid in \mathbb{R}^d . Hence the generic rigidity can be considered as a property

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 Japan, +81-75-753-7793, tanigawa@kurims.kyoto-u.ac.jp Supported by JSPS Grant-in-Aid for Scientific Research(A)(25240004) and JSPS Grant-in-Aid for Young Scientist (B), 24740058.

of the underlying graph. We say that a graph is *rigid* in \mathbb{R}^d if some/any generic realization of G is rigid in \mathbb{R}^d [1]. Laman's theorem [25] gives a combinatorial characterization of rigid graphs in \mathbb{R}^2 , which states that G is rigid in \mathbb{R}^2 if and only if G contains a subgraph H satisfying $|E(H)| = 2|V(G)| - 3$ and $|E(H')| \leq 2|V(H')| - 3$ for any subgraph H' of H with $|V(H')| \geq 2$. Extending Laman's theorem to rigidity in \mathbb{R}^3 is one of the most important open problems in this field. See, e.g., [34] for more details.

The analysis of global rigidity turns out to be more challenging, but substantial progress has been made in the last decade. The main tool for analyzing global rigidity is a stress matrix introduced by Connelly [5]. (For the definition of stress matrices, see, e.g., [6].) Connelly [6] showed that, if (G, \mathbf{p}) is generic and the rank of a stress matrix of (G, \mathbf{p}) is $|V(G)| - d - 1$, then (G, \mathbf{p}) is globally rigid. He also conjectured that the rank condition is necessary. This conjecture was recently confirmed by Gortler, Healy, and Thurston [13]. An important corollary of their result is:

Theorem 1.1 (Gortler, Healy, and Thurston [13]). *If there is a globally rigid generic realization of G in \mathbb{R}^d , then every generic realization of G is globally rigid in \mathbb{R}^d .*

Thus generic global rigidity is a property of graphs, and we can say that a graph G is *globally rigid* in \mathbb{R}^d if some/any generic realization of G is globally rigid.

Another breakthrough in this context is a combinatorial characterization of globally rigid graphs in \mathbb{R}^2 by Jackson and Jordán [16]. In [15] Hendrickson showed the following necessary condition for global rigidity of graphs.

Theorem 1.2 (Hendrickson [15]). *Let (G, \mathbf{p}) be a generic framework in \mathbb{R}^d . If (G, \mathbf{p}) is globally rigid, then either G is a complete graph on at most $d + 1$ vertices or G is $(d + 1)$ -connected and redundantly rigid, where G is called *redundantly rigid* if $G - e$ is rigid for all $e \in E(G)$.*

A useful method for generating rigid/globally rigid graphs is the so-called *Henneberg operation*, that is, an operation that creates a new graph by adding a new vertex. For a graph $G = (V, E)$, a *0-extension* in \mathbb{R}^d adds a new vertex with d new edges incident to the new vertex, and a *1-extension* in \mathbb{R}^d removes an existing edge e and adds a new vertex with $d + 1$ new incident edges so that the new vertex is incident with the both endvertices of e . It is known that both operations preserve the rigidity of graphs in \mathbb{R}^d , which gives a proof of Laman's theorem in case of $d = 2$, see [25]. Analogously, Connelly showed that the 1-extension preserves the corank of stress matrices and conjectured that for $d = 2$ every graph satisfying Hendrickson's condition can be generated from K_4 by a sequence of 1-extensions and edge additions.

Theorem 1.3 (Connelly [6], Jackson, Jordán and Szabadka [17], Szabadka [27]). ¹ *A graph obtained from a globally rigid graph in \mathbb{R}^d by a 1-extension is globally rigid in \mathbb{R}^d .*

The conjecture was confirmed by Berg and Jordán [2] for rigidity circuits and later by Jackson and Jordán [16] for the general case.

¹What Connelly proved in [6] is that 1-extension preserves the corank of stress matrices, which implies that 1-extension preserves the global rigidity due to the algebraic characterization by Gortler et al. [13]. The present form of Theorem 1.3 for $d = 2$ was first proved by Jackson, Jordán and Szabadka [17] before [13]. Their proof was extended for general dimension in [27]. Szabadka's result is actually more general, see the discussion after Lemma 4.1 for more detail.

Theorem 1.4 (Jackson and Jordán [16]). *A 3-connected redundantly rigid graph in \mathbb{R}^2 can be constructed from K_4 by a sequence of 1-extensions and edge additions.*

Combining these results, we have:

Theorem 1.5 (Connelly [6], Jackson and Jordán [16]). *A graph G is globally rigid in \mathbb{R}^2 if and only if G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid in \mathbb{R}^2 .*

Hendrickson [15] conjectured the converse of Theorem 1.2, and Theorem 1.5 confirms Hendrickson’s conjecture in $d = 2$. On the other hand Connelly pointed out that the conjecture is false for $d \geq 3$. In this paper we show that, if G is *vertex-redundantly rigid* in \mathbb{R}^d , then G is globally rigid in \mathbb{R}^d , where G is said to be vertex-redundantly rigid if $G - v$ is rigid for all $v \in V(G)$.

More generally, in this paper, we shall investigate how to find globally rigid generic realizations of graphs in \mathbb{R}^d based on elementary geometric observations. Our arguments lead to new proofs of combinatorial characterizations of the global rigidity of graphs in \mathbb{R}^2 (Theorem 1.4) and that of body-bar graphs in \mathbb{R}^d recently shown by Connelly, Jordán, and Whiteley [7] (Theorem 3.2). We also give extensions of the 1-extension theorem (Theorem 1.3) and Connelly’s composition theorem [8], which are main tools for generating globally rigid graphs in \mathbb{R}^d , and prove a conjecture by Frank and Jiang [11, Conjecture 29] on the global rigidity of k -chains.

Before closing the introduction, we should emphasize that Theorems 1.3, 1.4, 1.5 were proved before the work of Gortler et al. (Theorem 1.1). In particular, the result by Connelly [6], Jackson and Jordán [16] implies not only the existence of a globally rigid generic realization of a 3-connected redundantly rigid graph but also Theorem 1.1 for $d = 2$. Later, Jackson, Jordán and Szabadka [17] also gave a proof of Theorem 1.3 for $d = 2$ without using stress matrices. In this sense, their results lead to a proof of Theorem 1.1 which relies on deep graph-theoretical observations, which is still important as the proof of Theorem 1.1 in [13] mainly consists of geometric/algebraic observations.

Connelly’s stress matrix condition or the technique by Jackson, Jordán and Szabadka [17] can verify the global rigidity of *all* generic realizations of a graph. However, with the aid of Theorem 1.1, we may now turn our attention to finding *a* globally rigid generic realization of a graph. This paper shows that there are several simple techniques to find a generic globally rigid realization, which sometimes lead to much simpler arguments for proving a sufficiency of the global rigidity of graphs.

The paper is organized as follows. In Section 2 we give preliminary facts on rigidity. In Section 3, we shall show how to find a globally rigid generic realization of a body-bar graph. In Section 4, we shall discuss generalizations of the 1-extension theorem and the composition theorem and give applications. In Section 5, we shall discuss another application of a result in Section 4 to the global rigidity of body-hinge frameworks, which are important special cases of bar-joint frameworks which appear in many applications.

2 Preliminaries

For a graph $G = (V, E)$, we consider a smooth function $f_G : \mathbb{R}^{dV} \rightarrow \mathbb{R}^E$ defined by

$$f_G(\mathbf{q})(uv) = \|q_u - q_v\|^2 \quad (\mathbf{q} \in \mathbb{R}^{dV}, uv \in E). \quad (2)$$

The function is known as the *rigidity map* of G . The Jacobian of f_G at $\mathbf{p} \in \mathbb{R}^{dV}$ is called the *rigidity matrix* of (G, \mathbf{p}) , denoted by $R(G, \mathbf{p})$.

A vector in the kernel of $R(G, \mathbf{p})$ can be considered as an assignment $\mathbf{m} : V \rightarrow \mathbb{R}^d$, which is called an *infinitesimal motion* of (G, \mathbf{p}) . It turns out that, for any $d \times d$ skew-symmetric matrix S and $t \in \mathbb{R}^d$, an assignment $\mathbf{m} : V \rightarrow \mathbb{R}^d$ defined by $\mathbf{m}(v) = Sp_v + t$ is an infinitesimal motion of (G, \mathbf{p}) . Such an infinitesimal motion is called *trivial*. We say that (G, \mathbf{p}) is *infinitesimally rigid* if any infinitesimal motion of (G, \mathbf{p}) is trivial. If \mathbf{p} is generic, (G, \mathbf{p}) is rigid if and only if (G, \mathbf{p}) is infinitesimally rigid [1].

Note that there is a one-to-one correspondence between the set of rows of $R(G, \mathbf{p})$ and E . Hence, by using the row independence, one can define a matroid on E , which is called the *rigidity matroid* of (G, \mathbf{p}) . Since the rank of $R(G, \mathbf{p})$ is invariant from \mathbf{p} as long as \mathbf{p} is generic, the rigidity matroid of (G, \mathbf{p}) is the same as that of (G, \mathbf{q}) for any generic \mathbf{q} . Thus we define the *generic rigidity matroid* $\mathcal{R}_d(G)$ as the rigidity matroid of (G, \mathbf{p}) for any generic $\mathbf{p} \in \mathbb{R}^{dV}$. A graph $G = (V, E)$ is rigid if and only if G is a complete graph or the rank of $\mathcal{R}_d(G)$ is equal to $d|V| - \binom{d+1}{2}$.

A graph $G = (V, E)$ is called a *rigidity circuit* in \mathbb{R}^d if E is a circuit in $\mathcal{R}_d(G)$. By Laman's theorem, G is a rigidity circuit in \mathbb{R}^2 if and only if $|E| = 2|V| - 2$ and $|E(H)| \leq 2|V(H)| - 3$ for any proper subgraph H of G .

Let $[\mathbf{q}]$ be the equivalence class of \mathbf{q} in \mathbb{R}^{dV} defined by the congruence \equiv . For a framework (G, \mathbf{p}) , we denote by $c(G, \mathbf{p})$ the number of distinct classes $[\mathbf{q}]$ with $(G, \mathbf{q}) \simeq (G, \mathbf{p})$. The following elementary observation is a folklore, which turns out to be a key to the subsequent arguments. See, e.g., [13, 20, 27] for more detailed discussion.

Proposition 2.1. *Let (G, \mathbf{p}) be a generic realization of a graph G . If (G, \mathbf{p}) is rigid, then $c(G, \mathbf{p})$ is finite.*

Throughout the paper we shall use the following notation. For a point $p \in \mathbb{R}^d$ and $\epsilon \in \mathbb{R}$, let $B(p, \epsilon) = \{x \in \mathbb{R}^d \mid \|p - x\| < \epsilon\}$ and $\partial B(p, \epsilon) = \{x \in \mathbb{R}^d \mid \|p - x\| = \epsilon\}$.

For a graph $G = (V, E)$, let $N_G(v) \subseteq V \setminus \{v\}$ be the set of neighbors of a vertex v and let $d_G(v) = |N_G(v)|$. For a finite set X , let $K(X)$ be the edge set of the complete graph on X . For disjoint finite sets X and Y , let $K(X, Y)$ be the edge set of the complete bipartite graph on the vertex classes X and Y .

A *k-separation* of a graph G is a pair (G_1, G_2) of edge-disjoint subgraphs of G each with at least $k + 1$ vertices such that $G = G_1 \cup G_2$ and $|V(G_1) \cap V(G_2)| = k$. A graph G is said to be *k-connected* if it has at least $k + 1$ vertices and has no *j-separation* for all $0 \leq j \leq k - 1$. For a connected graph $G = (V, E)$, a vertex set $X \subseteq V$ with $|X| = k$ is called a *k-separator* if $G - X$ is disconnected.

3 Global Rigidity of Body-bar Frameworks

A body-bar framework is a structure consisting of rigid bodies linked by disjoint bars. By regarding each body as a dense bar-joint framework, one can consider a body-bar framework as a special case of bar-joint framework. The underlying graph of a body-bar framework is a multigraph obtained by identifying each body with a vertex and each bar with an edge.

Given a multigraph H which is the underlying graph of a body-bar framework in \mathbb{R}^d , the associated *body-bar graph* is defined by “replacing” each vertex v by a complete

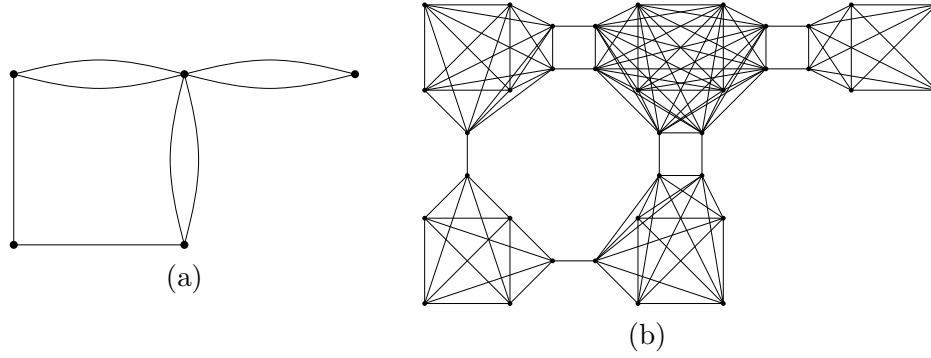


Figure 1: A graph H and its body-bar graph G_H for $d = 3$.

subgraph with $(d + 1 + d_H(v))$ vertices. Formally, the *body-bar graph* of H is a simple graph $G_H = (V_H, E_H)$, where

- V_H is the union of disjoint vertex sets B_H^v for each $v \in V(H)$ which is defined by $B_H^v = \{v_1^0, \dots, v_{d+1}^0\} \cup \{v_e^1 \mid e \in E(H) \text{ is incident to } v\}$;
- $E_H = (\bigcup_{v \in V} K(B_H^v)) \cup \{u_e^1 v_e^1 \mid e = uv \in E(H)\}$.

See Figure 1 for an example.

Note that B_H^v induces a complete subgraph in G_H , which is called a *body* associated with v . Each body forms a globally rigid subframework when realizing G_H generically. A bar-joint framework (G_H, \mathbf{p}) with $\mathbf{p} : V_H \rightarrow \mathbb{R}^d$ is called a *body-bar realization* of H in \mathbb{R}^d . A celebrated theorem by Tay gives a combinatorial characterization of generically rigid realizability.

Theorem 3.1 (Tay [30]). *Let H be a finite multigraph. Then there is a rigid and generic body-bar realization of H in \mathbb{R}^d if and only if H contains $\binom{d+1}{2}$ edge-disjoint spanning trees.*

Connelly, Jordán and Whiteley [7] recently proved the global rigidity counterpart of Tay's theorem, which confirms that Hendrickson's conjecture is true in this model.

Theorem 3.2 (Connelly, Jordán and Whiteley [7]). *Let H be a multigraph. Then there is a globally rigid and generic body-bar realization of H in \mathbb{R}^d if and only if there is a rigid and generic body-bar realization of $H - e$ in \mathbb{R}^d for every $e \in E(H)$.*

Proof. The necessity follows from Hendrickson's theorem (Theorem 1.2) and our target is to prove the sufficiency. The proof is done by induction on $|V(H)|$. The statement is clear if $|V(H)| = 1$, so assume $|V(H)| > 1$.

Let us take any $e = uv \in E(H)$ and let $H' = H - e$. Since $G_{H'}$ is rigid, by Proposition 2.1 there is a generic and rigid body-bar realization $(G_{H'}, \mathbf{p}')$ such that $c(G_{H'}, \mathbf{p}')$ is finite. Let us consider the set of equivalent realizations, that is,

$$\mathcal{E}(G_{H'}, \mathbf{p}') = \{\mathbf{q}' : V_{H'} \rightarrow \mathbb{R}^d \mid (G_{H'}, \mathbf{q}') \simeq (G_{H'}, \mathbf{p}'), \forall w \in B_{H'}^u, \mathbf{q}'(w) = \mathbf{p}'(w)\}$$

(where we restrict our attention to realizations fixing the body of u). Then $\mathcal{E}(G_{H'}, \mathbf{p}')$ is a finite set. Note that the body of v is a complete subgraph, and hence for each

$\mathbf{q}' \in \mathcal{E}(G_{H'}, \mathbf{p}')$ there is an isometry which can be written as a pair of a $d \times d$ orthogonal matrix $A_{\mathbf{q}'}$ and $t_{\mathbf{q}'} \in \mathbb{R}^d$ such that $\mathbf{q}'(w) = A_{\mathbf{q}'}\mathbf{p}'(w) + t_{\mathbf{q}'}$ for all $w \in B_{H'}^v$.

We now extend $\mathbf{p}' : V_{H'} \rightarrow \mathbb{R}^d$ to $\mathbf{p} : V_H \rightarrow \mathbb{R}^d$ as follows. Recall that $V_H \setminus V_{H'} = \{u_e^1, v_e^1\}$. We take $\mathbf{p}(u_e^1)$ and $\mathbf{p}(v_e^1)$ in such a way that $\{\mathbf{p}(u_e^1), \mathbf{p}(v_e^1)\}$ is algebraically independent over the field generated by \mathbb{Q} and the entries of $A_{\mathbf{q}'}$ and $t_{\mathbf{q}'}$ for all $\mathbf{q}' \in \mathcal{E}(G_{H'}, \mathbf{p}')$. Since $\mathcal{E}(G_{H'}, \mathbf{p}')$ is finite, such \mathbf{p} exists.

We prove that (G_H, \mathbf{p}) is globally rigid in \mathbb{R}^d . To see this let us consider

$$\mathcal{E}(G_H, \mathbf{p}) = \{\mathbf{q} : V_H \rightarrow \mathbb{R}^d \mid (G_H, \mathbf{q}) \simeq (G_H, \mathbf{p}), \forall w \in B_H^u, \mathbf{q}(w) = \mathbf{p}(w)\}.$$

Since $(G_{H'}, \mathbf{p}')$ is a subframework of (G_H, \mathbf{p}) , for each $\mathbf{q} \in \mathcal{E}(G_H, \mathbf{p})$ there is some $\mathbf{q}' \in \mathcal{E}(G_{H'}, \mathbf{p}')$ such that $\mathbf{q}(w) = A_{\mathbf{q}'}\mathbf{p}'(w) + t_{\mathbf{q}'}$ for all $w \in B_H^v$. In particular, $\mathbf{q}(v_e^1) = A_{\mathbf{q}'}\mathbf{p}'(v_e^1) + t_{\mathbf{q}'}$ holds by $v_e^1 \in B_H^v$. Moreover $\mathbf{p}(u_e^1) = \mathbf{q}(u_e^1)$ and $\|\mathbf{p}(u_e^1) - \mathbf{p}(v_e^1)\| = \|\mathbf{q}(u_e^1) - \mathbf{q}(v_e^1)\|$ due to the existence of edge $u_e^1 v_e^1$. Hence,

$$\|\mathbf{p}(u_e^1) - \mathbf{p}(v_e^1)\|^2 - \|\mathbf{p}(u_e^1) - A_{\mathbf{q}'}\mathbf{p}'(v_e^1) - t_{\mathbf{q}'}\|^2 = 0.$$

If we regard the left hand side as a polynomial with variables $\mathbf{p}(u_e^1)$ and $\mathbf{p}(v_e^1)$, then this polynomial is not identically zero unless $A_{\mathbf{q}'}$ is the identity matrix and $t_{\mathbf{q}'} = 0$. (For example, take $\mathbf{p}(u_e^1) = \mathbf{p}(v_e^1) = p$ for any $p \in \mathbb{R}^d$ with $(I - A_{\mathbf{q}'})p \neq t_{\mathbf{q}'}$.) Thus, due to the choice of $\mathbf{p}(u_e^1)$ and $\mathbf{p}(v_e^1)$, $A_{\mathbf{q}'}$ is the identity and $t_{\mathbf{q}'} = 0$.

Consequently, for any $\mathbf{q} \in \mathcal{E}(G_H, \mathbf{p})$, we have $\mathbf{q}(w) = \mathbf{p}(w)$ for all $w \in B_H^u \cup B_H^v$, and hence the subgraph induced by $B_H^u \cup B_H^v$ can be regarded as one body; More precisely, there is a globally rigid generic body-bar realization of H if and only if there is a globally rigid generic body-bar realization of H^* , where H^* is the graph obtained from H by contracting u and v . Clearly there is a generic rigid body-bar realization of $H^* - f$ for all $f \in E(H^*)$, and hence there is a globally rigid generic body-bar realization of H^* by induction. This completes the proof. \square

Remark. Theorem 3.2 along with Theorem 1.1 and Theorem 3.1 implies a combinatorial characterization of the global rigidity of body-bar graphs, which is the main theorem of [7]. We should remark that the proof in [7] is done by the evaluation of the rank of stress matrices by using a constructive characterization of Frank and Szegő [10]. Thus their proof further implies Theorem 1.1 for body-bar graphs.

Remark. The definition of body-bar graphs in [7] is slightly different from the one given above. In [7] each vertex of H is replaced by a complete graph on $d_H(v)$ vertices. One can easily check that this distinction does not cause any difference of the global rigidity property.

4 Global Rigidity of Bar-joint Frameworks

In this section we shall discuss the global rigidity of generic bar-joint frameworks in \mathbb{R}^d .

4.1 Vertex Removal Lemma

The following lemma extends Theorem 1.3, which turns out to be a very powerful tool as shown in applications in subsequent sections.

Lemma 4.1. *Let $G = (V, E)$ be a graph and v be a vertex of degree more than d . Suppose that*

- $G - v$ is rigid in \mathbb{R}^d , and
- $G - v + K(N_G(v))$ is globally rigid in \mathbb{R}^d .

Then G is globally rigid in \mathbb{R}^d .

Proof. Let $(G - v, \mathbf{p}')$ be a generic realization of $G - v$, and let us take any pair of distinct vertices i, j in $N_G(v)$ such that $ij \notin E(G)$. Since $G - v$ is rigid, we may assume that $c(G - v, \mathbf{p}')$ is finite by Proposition 2.1. Hence, the number of possible distances between q'_i and q'_j over all $\mathbf{q}' : V(G - v) \rightarrow \mathbb{R}^d$ with $(G - v, \mathbf{q}') \simeq (G - v, \mathbf{p}')$ is finite. This implies that there is $\epsilon > 0$ such that, for any \mathbf{q}' with $(G - v, \mathbf{q}') \simeq (G - v, \mathbf{p}')$, $||p'_i - p'_j|| - ||q'_i - q'_j|| \leq \epsilon$ implies $||q'_i - q'_j|| = ||p'_i - p'_j||$.

Let us extend \mathbf{p}' to $\mathbf{p} : V(G) \rightarrow \mathbb{R}^d$ such that \mathbf{p} is generic and $p_v \in B(p_i, \frac{\epsilon}{4})$.

Consider any $\mathbf{q} : V(G) \rightarrow \mathbb{R}^d$ with $(G, \mathbf{q}) \simeq (G, \mathbf{p})$. By $p_v \in B(p_i, \frac{\epsilon}{4})$ and $i, j \in N_G(v)$, we have $||p_i - p_j|| - ||q_i - q_j|| \leq \epsilon$. Since $(G - v, \mathbf{p}')$ is a subframework of (G, \mathbf{p}) , it follows that $||q_i - q_j|| = ||p_i - p_j||$. Therefore, (G, \mathbf{p}) is globally rigid if $(G + ij, \mathbf{p})$ is globally rigid. In other words G is globally rigid if (and only if) $G + ij$ is globally rigid.

Applying the same argument to $G + ij$ and then repeatedly to the resulting larger graphs we conclude that G is globally rigid if and only if $G + K(N_G(v))$ is globally rigid. The global rigidity of $G + K(N_G(v))$ now follows from the global rigidity of $G - v + K(N_G(v))$ since v has degree more than d . \square

Remark. Lemma 4.1 is also implicit in [27, Lemma 2.3.1] with a completely different argument. It should be noted that Szabadka's proof does not rely on Theorem 1.1. See also [17, 20].

Our proof of Lemma 4.1 relies on Theorem 1.1. In the appendix of the preprint version of this paper [29] we show how to find a generic globally rigid realization based on elementary geometric observations without using Theorem 1.1 in $d \leq 3$.

4.2 Applications

In this subsection we shall give two applications of Lemma 4.1.

Theorem 4.2. *Let $G = (V, E)$ be a vertex-redundantly rigid graph in \mathbb{R}^d . Then, G is globally rigid in \mathbb{R}^d .*

Proof. The proof is done by induction on $|V|$. When $|V| \leq d + 2$, a vertex-redundantly rigid graph G is isomorphic to $K_{|V|}$, which is globally rigid in \mathbb{R}^d by definition.

Let us assume $|V| > d + 2$. Take any vertex $v \in V$. If $d_G(v) \leq d$, then the removal of any neighbor of v results in a flexible framework. Hence, $d_G(v) \geq d + 1$ due to the vertex-redundancy of G . Also $G - v$ is rigid.

Let $G' = G - v + K(N_G(v))$. Suppose that G' is not vertex-redundantly rigid. Then there is a vertex u such that $G' - u$ is not rigid. Since $d_G(v) \geq d + 1$, $N_G(v) \setminus \{u\}$ induces a complete subgraph of at least d vertices in $G' - u$. Therefore, $G - u$ cannot be rigid, which contradicts the vertex-redundancy of G . Thus G' is vertex-redundantly rigid, and by induction G' is globally rigid in \mathbb{R}^d . By Lemma 4.1 G is also globally rigid in \mathbb{R}^d . \square

The extremal properties of vertex-redundantly rigid graphs in \mathbb{R}^d are investigated in [26, 23], where vertex-redundantly rigid graphs are referred to as bi-rigid graphs.

We also remark that the characterization for body-bar frameworks (Theorem 3.2) easily follows from Theorem 4.2.

Next we consider the global rigidity of k -chains. Following Frank and Jiang [11], we define a k -chain as a bipartite graph on disjoint k sets A_1, A_2, \dots, A_k such that $A_i \cup A_{i+1}$ induces the complete bipartite graph on A_i and A_{i+1} for all $1 \leq i \leq k-1$ and there are no other edges. Note that if $k = 2$ or $k = 3$ a k -chain is a complete bipartite graph. Frank and Jiang [11] gave a characterization of the global rigidity of k -chains with $\binom{d+1}{2}$ vertices and conjectured the case of more than $\binom{d+1}{2}$ vertices [11, Conjecture 29]. Here we give an affirmative answer to their conjecture.

Theorem 4.3. *Any $(d+1)$ -connected k -chain with more than $\binom{d+1}{2}$ vertices is globally rigid in \mathbb{R}^d .*

Proof. By Bolker and Roth [3], it is known that any $(d+1)$ -connected complete bipartite graph is rigid in \mathbb{R}^d if the number of vertices is at least $\binom{d+1}{2}$. Frank and Jiang showed that any $(d+1)$ -connected k -chain is rigid if $k \geq 4$ and the number of vertices is $\binom{d+1}{2}$ [11, Lemma 17]. We claim that any $(d+1)$ -connected k -chain G with at least $\binom{d+1}{2}$ vertices is rigid. This can be checked as follows. If $k \leq 3$ the claim follows from the result of Bolker and Roth. If $k > 3$, then we can take $X_i \subseteq A_i$ with $|X_i| = d+1$ for $i = 2, 3$. By inductively adding vertices to the induced subgraph by $X_2 \cup X_3$, one can find a $(d+1)$ -connected k' -chain G' in G with $\binom{d+1}{2}$ vertices for some $k' \leq k$. This subgraph G' is rigid by the result of Frank and Jiang, and G is rigid since G can be obtained from G' by inductively adding the vertices along with at least d edges.

Now consider any $(d+1)$ -connected k -chain G and without loss of generality assume $|A_1| \leq |A_k|$. Let us take a vertex v from A_1 . Then $G-v$ is still a $(d+1)$ -connected k -chain and hence is rigid. Also $G-v+K(N_G(v))$ is globally rigid in \mathbb{R}^d , since $K(N_G(v)) = K(A_2)$ is globally rigid by $|A_2| \geq d+1$ and $G-v+K(N_G(v))$ can be obtained from $K(A_2)$ by inductively adding vertices along with at least $d+1$ edges.

Therefore Lemma 4.1 implies that G is globally rigid in \mathbb{R}^d . \square

4.3 A Proof of Theorem 1.5

In this section we shall give a new proof of the sufficiency of Theorem 1.5.

Since Lemma 4.1 is an extension of 1-extension theorem (Theorem 1.3), it might be possible to replace Theorem 1.4 with a shorter combinatorial argument for proving Theorem 1.5. Here we show that the following weaker version of Theorem 1.4 suffices for proving Theorem 1.5.

Lemma 4.4. *Let G be a 3-connected redundantly rigid graph in \mathbb{R}^2 . Then G has a vertex of degree three or has an edge e for which $G-e$ is 3-connected and redundantly rigid.*

Before showing Lemma 4.4 let us give a proof of the sufficiency of Theorem 1.5.

Proof of the sufficiency of Theorem 1.5. The proof is done by induction on the lexicographical order of $(|V|, |E|)$. The base case is when G is isomorphic to K_4 , which is globally rigid.

Let us consider the case when $|V| > 4$. By Lemma 4.4, G has a vertex v of degree three or has an edge e for which $G - e$ is 3-connected and redundantly rigid in \mathbb{R}^2 . If G has such an edge e , then by induction there is a globally rigid generic realization $(G - e, \mathbf{p})$, and (G, \mathbf{p}) is globally rigid. If G has a vertex of degree three, then $G - v$ is rigid and $G - v + K(N_G(v))$ is 3-connected and redundantly rigid since G is 3-connected and redundantly rigid. Therefore by Lemma 4.1 G is globally rigid in \mathbb{R}^2 . \square

The remaining of this subsection is devoted to the proof of Lemma 4.4. For proving it (without using Theorem 1.4), we still need several observations given by Berg and Jordán [2] and Jackson and Jordán [16] at an early stage of the proof of Theorem 1.4. Some of these observations are already nontrivial, but reasonably short proofs were given there.

To explain these ingredients we need some terminology. Let \mathcal{M} be a matroid on a finite set E . We define a relation \sim on E such that $e \sim e'$ if there is a circuit of \mathcal{M} that contains e and e' . The relation \sim is known to be an equivalence relation, and an equivalence class with respect to \sim is called an *connected component* of \mathcal{M} . A subset $F \subseteq E$ is called *connected* if $e \sim e'$ for any $e, e' \in F$, and \mathcal{M} is called *connected* if E is connected.

A sequence C_1, C_2, \dots, C_t of circuits in \mathcal{M} is called a *partial ear decomposition* of \mathcal{M} if for any $2 \leq i \leq t$

- $C_i \cap D_{i-1} \neq \emptyset$, where $D_i = C_1 \cup C_2 \cup \dots \cup C_i$;
- $C_i \setminus D_{i-1} \neq \emptyset$;
- for any circuit C with $C \cap D_{i-1} \neq \emptyset$ and $C \setminus D_{i-1} \neq \emptyset$, $C \setminus D_{i-1} \subset C_i \setminus D_{i-1}$ does not hold.

A partial ear-decomposition is called an *ear decomposition* if $\bigcup_{i=1}^t C_i = E$.

Theorem 4.5 (Coullard and Hellerstein [4]). *A matroid \mathcal{M} is connected if and only if \mathcal{M} has an ear decomposition. If \mathcal{M} is connected, then any partial ear decomposition can be extended to an ear decomposition.*

Now let us come back to rigidity. Following [16], we say that a graph G is *M-connected* in \mathbb{R}^d if $\mathcal{R}_d(G)$ is connected.

Theorem 4.6 ([16], Theorem 3.2). *Suppose that a graph G is 3-connected. Then G is redundantly rigid in \mathbb{R}^2 if and only if G is M-connected in \mathbb{R}^2 .*

The following lemma is implicit in [16, Lemma 5.2] and explicit in [21].

Lemma 4.7. *Let G be an M-connected graph in \mathbb{R}^2 , C_1, C_2, \dots, C_t be an ear decomposition of $\mathcal{R}_2(G)$, and $D_i = \bigcup_{j=1}^i C_j$ where $D_0 = \emptyset$. If $V(C_t) \subseteq V(D_{t-1})$, then $|C_t| = 1$; otherwise G has a vertex of degree three that is contained in $V(C_t) \setminus V(D_{t-1})$.*

We also need certain properties of rigid graphs related to 2-separators. We say that two 2-separators $\{u, v\}$ and $\{s, t\}$ in G *cross* if s and t are in distinct connected components of $G - u - v$.

Lemma 4.8 ([16], Lemma 3.6). *If G is rigid in \mathbb{R}^2 , then any two 2-separators do not cross.*

Let G_1 and G_2 be graphs with $V(G_1) \cap V(G_2) = \emptyset$. For edge $a_1b_1 \in E(G_1)$ and $a_2b_2 \in E(G_2)$, the 2-sum $G_1 \oplus_2 G_2$ of G_1 and G_2 (along a_1b_1 and a_2b_2) is the graph obtained from $G_1 - a_1b_1$ and $G_2 - a_2b_2$ by identifying a_1 with a_2 and b_1 with b_2 .

Conversely, suppose that (G_1, G_2) is a 2-separation of a graph G with $V(G_1) \cap V(G_2) = \{a, b\}$. For $i = 1, 2$, let $G'_i = G_i + ab$ if $ab \notin E(G_i)$ and otherwise $G'_i = G_i$. We say that G'_1 and G'_2 are the *cleavage graphs* obtained by *cleaving* G along $\{a, b\}$.

Lemma 4.9 ([16], Lemma 3.3). *Suppose G_1 and G_2 are M -connected in \mathbb{R}^2 . Then the 2-sum $G_1 \oplus_2 G_2$ is M -connected in \mathbb{R}^2 .*

Lemma 4.10 ([16], Lemma 3.4). *Suppose G_1 and G_2 are obtained from a M -connected graph G in \mathbb{R}^2 by cleaving G along a 2-separator. Then G_1 and G_2 are M -connected in \mathbb{R}^2 .*

For a 2-connected graph $G = (V, E)$, a nonempty set $X \subset V$ is called a *fragment* if $|N_G(X)| = 2$ and $V \setminus (X \cup N_G(X)) \neq \emptyset$. Now we are ready to prove Lemma 4.4.

Proof of Lemma 4.4. By Theorem 4.6 G is M -connected. Suppose that G has no vertex of degree three. By Lemma 4.7, for any ear decomposition C_1, \dots, C_t of G , $V(C_t) \subseteq V(D_{t-1})$, where $D_{t-1} = \bigcup_{i=1}^{t-1} C_i$. Hence, $|C_t| = 1$ by Lemma 4.7, and $G - e$ is M -connected for $e \in C_t$.

Let $E' = \{e \in E(G) \mid G - e \text{ is } M\text{-connected}\}$, which is nonempty as we have just noted. If $G - e$ is 3-connected for some $e \in E'$, then we are done. Hence we assume that $G - e$ is not 3-connected for any $e \in E'$.

Let us take $e \in E'$ such that the size of the smallest fragment in $G - e$ is minimized over all $e \in E'$. Let (H_1, H_2) be the 2-separator of $G - e$ such that $V(H_1) \setminus V(H_2)$ is the smallest fragment, and denote $\{a, b\} = V(H_1) \cap V(H_2)$. By Lemma 4.10, the graphs H'_1, H'_2 obtained by cleaving along ab are M -connected.

Let v be the endvertex of e in H'_1 . Let us take a circuit C of $\mathcal{R}_2(H'_1)$ such that $ab \in C$ and $v \in V(C)$. Such a circuit exists since H'_1 is M -connected. By Theorem 4.5, there is an ear decomposition C'_1, \dots, C'_s of H'_1 with $C'_1 = C$ for some $s \geq 1$.

If $s = 1$, H'_1 is a circuit and has a vertex of degree three incident to neither e nor ab , implying that G has a vertex of degree three, a contradiction. If $s > 1$, by Lemma 4.7 H'_1 has a vertex u of degree three with $u \in V(C'_s) \setminus V(C'_1)$ or an edge $f \in V(C'_s)$ such that $H'_1 - f$ is M -connected. In the former case u is incident to neither e nor ab , implying that G has a vertex of degree three, a contradiction. In the later case, since $H'_1 - f$ is M -connected, $G - f$ is M -connected by Lemma 4.9. Hence $G - f$ is not 3-connected, and there is a separator in $G - f$. However, since f connects two vertices in H_1 , Lemma 4.8 implies that a fragment in $G - f$ is properly contained in $V(H_1) \setminus V(H_2)$. This contradicts the choice of e . \square

4.4 Combining Two Graphs

We next consider another operation for constructing globally rigid graphs from smaller graphs. Let G be a graph, X be a subset of $V(G)$, and H be a graph on X (whose edges may not be in G). We say that (H, X) is a *rooted minor* of (G, X) if H can be obtained from G by deleting and contracting edges of G , i.e., there is a partition $\{U_v \mid v \in X\}$ of $V(G)$ into $|X|$ subsets, each of which is indexed by an element of X , such that (i) $v \in U_v$,

(ii) the induced subgraph of G by U_v is connected for each $v \in X$, and (iii) G has an edge between a vertex in U_u and a vertex in U_v for each $uv \in E(H)$.

Theorem 4.11. *Let G_1 and G_2 be graphs with $X = V(G_1) \cap V(G_2)$ and H be a graph on X whose edges may not be in $G_1 \cup G_2$. Suppose that*

- $|X| \geq d + 1$,
- G_1 is rigid in \mathbb{R}^d ,
- (H, X) is a rooted minor of (G_2, X) .
- $G_1 \cup H$ and $G_2 \cup K(X)$ are globally rigid or $G_1 \cup K(X)$ and $G_2 \cup H$ are globally rigid in \mathbb{R}^d .

Then $G_1 \cup G_2$ is globally rigid in \mathbb{R}^d .

Proof. Let $\mathbf{p} : V(G_1 \cup G_2) \rightarrow \mathbb{R}^d$ be generic, and \mathbf{p}_i be the restriction of \mathbf{p} to $V(G_i)$ for $i = 1, 2$. Since (H, X) is a rooted minor of (G_2, X) , there is a partition $\{U_v \mid v \in X\}$ of $V(G_2)$ into $|X|$ subsets, each of which is indexed by an element of X , such that, for each $xy \in E(H)$, there is a path P_{xy} from x to y such that P_{xy} is in the induced subgraph of G_2 by $U_x \cup U_y$ and passes through an edge between U_x and U_y exactly once.

We shall assume that, for each $w \in U_v$, $p_w \in B(p_v, \epsilon)$ holds for some $\epsilon > 0$. Due to the existence of P_{xy} for each $xy \in E(H)$, there is a constant C such that, for any \mathbf{q}_2 with $(G_2, \mathbf{q}_2) \simeq (G_2, \mathbf{p}_2)$,

$$\|p_x - p_y\| - \|q_x - q_y\| \leq C|V(G_2)|\epsilon \quad (3)$$

for all $xy \in E(H)$.

Since (G_1, \mathbf{p}_1) is rigid, $c(G_1, \mathbf{p}_1)$ is finite. Therefore, for each $xy \in E(H)$, there are finite number of possible distances between q_x and q_y over all \mathbf{q} with $(G_1, \mathbf{q}_1) \simeq (G_1, \mathbf{p}_1)$. Since the position of p_w for $w \in V(G_2) \setminus X$ is independent from these distances, if we take small enough ϵ , we have

$$\|p_x - p_y\| = \|q_x - q_y\| \quad \text{for } xy \in E(H)$$

for every \mathbf{q} with $(G_1 \cup G_2, \mathbf{q}) \simeq (G_1 \cup G_2, \mathbf{p})$. Therefore, $G_1 \cup G_2$ is globally rigid in \mathbb{R}^d if $G_1 \cup G_2 \cup H$ is globally rigid.

By assumption, for some $i \in \{1, 2\}$, $G_i \cup H$ is globally rigid. However, since $G_1 \cup G_2 \cup H$ contains $G_i \cup H$ as a subgraph and $(G_i \cup H, \mathbf{p}_i)$ is globally rigid in \mathbb{R}^d ,

$$\|p_v - p_w\| = \|q_u - q_w\| \quad \text{for all } u, w \in X$$

for every \mathbf{q} with $(G_1 \cup G_2 \cup H, \mathbf{q}) \simeq (G_1 \cup G_2 \cup H, \mathbf{p})$. Thus $G_1 \cup G_2 \cup H$ is globally rigid if $G_1 \cup G_2 \cup K(X)$ is globally rigid. Since $|X| \geq d + 1$ and $G_1 \cup K(X)$ and $G_2 \cup K(X)$ are globally rigid, $G_1 \cup G_2 \cup H$ is globally rigid. This in turn implies the global rigidity of (G, \mathbf{p}) . \square

Theorem 4.11 extends [8, Theorem 11], which copes with the case when $|X| = d + 1$, $|E(H)| = 1$, $G_i \cup H$ are globally rigid for each $i = 1, 2$. Indeed, in this case, $G_i \cup H$ is $(d + 1)$ -connected and redundantly rigid by Theorem 1.2. So G_i is rigid. Denoting the

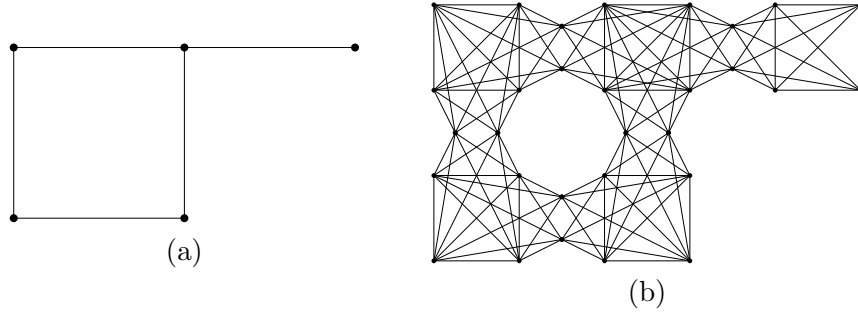


Figure 2: (a) H and (b) G_H in \mathbb{R}^3 .

edge in $E(H)$ by ab , $G_2 - (X \setminus \{a, b\})$ is connected. Hence (H, X) is a rooted minor of (G, X) .

Notice also that Theorem 4.11 contains Theorem 1.3 as a special case. In this sense, Theorem 4.11 is a common generalization of the 1-extension theorem and the composition theorem [8, Theorem 11] by Connelly.

5 Global Rigidity of Body-hinge Frameworks

In this section we consider body-hinge frameworks. A body-hinge framework is a structural model consisting of rigid bodies connected by hinges. A hinge is a $(d - 2)$ -dimensional affine space which links two bodies, and the bodies can rotate around the hinge.

Given a body-hinge framework, where each hinge connects two bodies, we shall define the underlying graph H by associating a vertex with a body and an edge with each hinge. As in the case of body-bar frameworks one can regard a body-hinge framework as a bar-joint framework by replacing each rigid body with a complete bar-joint framework. Since each hinge is a $(d - 2)$ -dimensional affine subspace, we are interested in the rigidity of the associated *body-hinge graph* $G_H = (V_H, E_H)$ defined as follows:

- $V_H = (\bigcup_{v \in V(H)} B_v) \cup (\bigcup_{e \in E(H)} H_e)$, where $B_v = \{v_1, \dots, v_{d+1}\}$ for each $v \in V(H)$ (which induces the body associated with v) and $H_e = \{h_{e,1}, h_{e,2}, \dots, h_{e,d-1}\}$ for each $e \in E(H)$ (which induces the hinge of e);
- $E_H = (\bigcup_{v \in V(H)} K(B_v)) \cup (\bigcup_{e \in E(H)} K(B_v, H_e))$, where $E(v)$ denotes the set of edges incident to v .

Figure 2 provides an example. A bar-joint framework (G_H, \mathbf{p}) with $\mathbf{p} : V_H \rightarrow \mathbb{R}^d$ is called a *body-hinge realization* of H in \mathbb{R}^d . We are interested in the rigidity/global rigidity of G_H for a given H .

For a graph H and a positive integer k , let kH denote the graph obtained from H by replacing each edge by k parallel copies. For each edge $e \in E(H)$, ke denote the set of the k copies of e in kH . Tay [31, 32] and Whiteley [33] independently gave a combinatorial characterization of the rigidity of body-hinge frameworks.

Theorem 5.1 (Tay [31, 32], Whiteley [33]). *Let G_H be the body-hinge graph of a graph H . Then G_H is rigid in \mathbb{R}^d if and only if $((\binom{d+1}{2} - 1)H$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees.*

Connelly and Whiteley [9] conjectured that $K_{5,5}$ is the only 4-connected redundantly rigid graph in \mathbb{R}^3 that is not globally rigid. Observe now that, for a graph H , if G_H is rigid, then G_H is 4-connected and redundantly rigid unless $|V(H)| = 1$. Therefore, a special case of the conjecture by Connelly and Whiteley states that G_H is rigid in \mathbb{R}^3 if and only if G_H is globally rigid in \mathbb{R}^3 . Recently, counterexamples for this special case were discovered [22], which also implies the existence of counterexamples for the Connelly-Whiteley conjecture.

Also in [22] a corrected combinatorial characterization of the global rigidity of body-hinge frameworks were also presented. Here we shall show that a slightly weaker statement follows from Theorem 4.2.

Theorem 5.2. *Let G_H be the body-hinge graph of a graph H . If $((\binom{d+1}{2} - 1)(H - e) + ((\binom{d+1}{2} - 3)e$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees for every $e \in E(H)$, then G_H is globally rigid in \mathbb{R}^d .*

Proof. By Theorem 4.2 it suffices to show that G_H is vertex-redundantly rigid. To this end, it suffices to show the rigidity of $G_H - h_{e,d-1}$ for each $e \in E(H)$ (where recall that $h_{e,d-1}$ is a vertex in V_H associated with e). Let $G_1 = G_H - h_{e,d-1}$.

Intuitively, (G_1, \mathbf{p}) is a framework in which bodies are linked by hinges except that the hinge associated with edge e is replaced with a $(d-3)$ -dimensional affine space (i.e., the bodies incident to the hinge of e are linked at points $p_{h_{e,1}}, \dots, p_{h_{e,d-2}}$). We now show that G_1 is rigid. The proof idea is essentially from [33, 19].

By the lemma assumption $((\binom{d+1}{2} - 1)(H - e) + ((\binom{d+1}{2} - 3)e$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees $T_{i,j}$, $1 \leq i < j \leq d+1$. Without loss of generality we may assume that

$$\left(\left(\left(\binom{d+1}{2} - 3 \right) e \right) \cap (T_{d-1,d} \cup T_{d-1,d+1} \cup T_{d,d+1}) = \emptyset. \quad (4)$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the standard basis of \mathbb{R}^d , and for simplicity of description the origin of \mathbb{R}^d is denoted by \mathbf{e}_{d+1} . We shall define $\mathbf{p} : V(G_H) \rightarrow \mathbb{R}^d$ as follows. For each $v_i \in B_v$ let $\mathbf{p}(v_i) = \mathbf{e}_i$. For each $f \in E(H) \setminus \{e\}$, there is at least one pair (k, l) of indices such that $((\binom{d+1}{2} - 1)f) \cap T_{k,l} = \emptyset$. Hence one can define $\mathbf{p}(h_{f,i})$ such that $\{\mathbf{p}(h_{f,i}) \mid 1 \leq i \leq d-1\} = \{\mathbf{e}_i \mid 1 \leq i \leq d+1, i \neq k, l\}$. Also, for e , we shall define $\mathbf{p}(h_{e,i})$ such that $\{\mathbf{p}(h_{f,i}) \mid 1 \leq i \leq d-2\} = \{\mathbf{e}_i \mid 1 \leq i \leq d-2\}$.

Now we show that (G_1, \mathbf{p}) is infinitesimally rigid in \mathbb{R}^d . To this end let us take any infinitesimal motion $\mathbf{m} : V(G_H) \rightarrow \mathbb{R}^d$ of (G_H, \mathbf{p}) . The proof is completed if we can show \mathbf{m} is trivial, i.e., there is a $d \times d$ skew-symmetric matrix S and $t \in \mathbb{R}^d$ such that $\mathbf{m}(v) = S\mathbf{p}(v) + t$ for each $v \in V_H$. Since $\{\mathbf{p}(v_i) \mid v_i \in B_v\}$ affinely spans \mathbb{R}^d and B_v induces the complete graph, for each $v \in V(H)$, there is a $d \times d$ skew-symmetric matrix S_v and $t_v \in \mathbb{R}^d$ such that $\mathbf{m}(v_i) = S_v\mathbf{p}(v_i) + t_v$ for every $v_i \in B_v$ and $\mathbf{m}(h_{f,j}) = S_v\mathbf{p}(h_{f,j}) + t_v$ for every $h_{f,j}$ incident to B_v .

For $1 \leq i < j \leq d+1$, consider any edge $f = uv \in T_{i,j}$. If $f \neq e$, by the definition of \mathbf{p} , there is at least one index k with $1 \leq k \leq d-1$ such that $\mathbf{p}(h_{f,k})$ is either \mathbf{e}_i or \mathbf{e}_j . Similarly, if $f = e$, there is k with $1 \leq k \leq d-2$ such that $\mathbf{p}(h_{f,k})$ is either \mathbf{e}_i or \mathbf{e}_j by (4). Therefore, one can take a vertex v_l from B_v such that $\{\mathbf{p}(h_{f,k}), \mathbf{p}(v_l)\} = \{\mathbf{e}_i, \mathbf{e}_j\}$.

Let us assume $\mathbf{p}(h_{f,k}) = \mathbf{e}_i$ and $\mathbf{p}(v_l) = \mathbf{e}_j$. The first-order length constraint by edge

$h_{f,k}v_l$ implies

$$\begin{aligned} 0 &= \langle \mathbf{p}(h_{f,k}) - \mathbf{p}(v_l), \mathbf{m}(h_{f,k}) - \mathbf{m}(v_l) \rangle \\ &= \langle \mathbf{p}(h_{f,k}) - \mathbf{p}(v_l), S_u \mathbf{p}(h_{f,k}) + t_u - S_v \mathbf{p}(v_l) - t_v \rangle \\ &= \langle \mathbf{e}_i - \mathbf{e}_j, S_u \mathbf{e}_i + t_u - S_v \mathbf{e}_j - t_v \rangle \\ &= -\mathbf{e}_j^\top S_u \mathbf{e}_i - \mathbf{e}_i^\top S_v \mathbf{e}_j + \langle \mathbf{e}_i - \mathbf{e}_j, t_u - t_v \rangle. \end{aligned} \quad (5)$$

This equation follows even when $\mathbf{p}(h_{f,k}) = \mathbf{e}_j$ and $\mathbf{p}(v_l) = \mathbf{e}_i$ by changing the role of u and v .

For $j = d + 1$, since $\mathbf{e}_{d+1} = 0$, (5) implies

$$\langle \mathbf{e}_i, t_u - t_v \rangle = 0 \quad \text{for } 1 \leq i \leq d \text{ and } uv \in T_{i,d+1}.$$

This implies $t_a = t_b$ for any pair $a, b \in V(H)$ since $T_{i,d+1}$ is a spanning tree. Therefore, using the skew-symmetry of S_v , (5) becomes

$$\mathbf{e}_i^\top S_v \mathbf{e}_j = \mathbf{e}_i^\top S_u \mathbf{e}_j \quad \text{for } 1 \leq i < j \leq d \text{ and } uv \in T_{i,j}.$$

This in turn implies $S_a = S_b$ for any pair $a, b \in V(H)$. Thus \mathbf{m} is trivial. \square

One important corollary of Theorem 5.2 is the following.

Corollary 5.3. *Let G_H be the body-hinge graph of a graph H . If G_H is hinge-redundantly rigid (i.e., G_{H-e} is rigid for any $e \in E(H)$), then G_H is globally rigid.*

It is also natural to consider a more general body-hinge model where each hinge may connect more than two bodies. Such a framework is called an *identified body-hinge framework*. The underlying graph becomes a hypergraph (or the corresponding bipartite graph), and a combinatorial characterization of the rigidity is known in terms of a count condition of the underlying graphs [31, 32, 28]. The argument can be easily generalized to show the counterpart of Corollary 5.3 for identified body-hinge frameworks.

A challenging open problem is to show a characterization for *panel-hinge frameworks* or *hinge-concurrent frameworks*, where in a panel-hinge framework the hinges incident to a body lie on a 2-dimensional affine space, and in a hinge-concurrent framework all the hinges incident to a body intersect at a point. It is known that in \mathbb{R}^3 the infinitesimal rigidity of such frameworks coincides with that of *molecular frameworks*, which are bar-joint frameworks whose underlying graphs are the square G^2 of graphs G . A combinatorial characterization of the generic rigidity of those frameworks was shown in [24].

6 Concluding Remarks

The global rigidity of graphs in complex spaces was analyzed by Jackson and Owen [20] and independently by Gortler and Thurston [14]. Bill Jackson pointed out that the proof of Theorem 3.2 can be applied even in the complex setting.

It is a long standing open problem whether there is a constant connectivity upper bound for the rigidity of graphs in \mathbb{R}^d for $d > 2$. Tibor Jordán pointed out that the existence of such a bound immediately implies a constant connectivity upper bound for the global rigidity by Theorem 4.2.

One of the challenging problems in this context is to determine which 4-connected redundantly rigid graphs are globally rigid in \mathbb{R}^3 . Currently it is not clear how large a subclass of 4-connected redundantly rigid graphs we can construct by the operations of Lemma 4.1 and Theorem 4.11.

Acknowledgment

We would like to thank Bill Jackson, Tibor Jordán, and referees for helpful comments and corrections. We would also like to thank Tibor Jordán for a discussion about applications of Theorem 4.2.

References

- [1] L. Asimov and B. Roth, The rigidity of graphs. *Tran. Amer. Math. Soc.*, 245:279–289, 1978.
- [2] A. Berg and T. Jordán, A proof of Connelly’s conjecture on 3-connected circuits of the rigidity matroid, *J. Combin. Theory Ser. B*, 88:77–97, 2003.
- [3] E. Bolker and B. Roth, When is a bipartite graph a rigid framework? *Pac. J. Math.*, 90:27–44, 1980.
- [4] C. Coullard and L. Hellerstein, Independence and port oracles for matroids, with an application to computational learning theory, *Combinatorica*, 16:189–208, 1996.
- [5] R. Connelly, Rigidity and energy, *Invent. Math.* 66:11–33, 1982.
- [6] R. Connelly, Generic global rigidity, *Discrete Comput. Geom.* 33:549–563, 2005.
- [7] R. Connelly, T. Jordán, and W. Whiteley, Generic global rigidity of body-bar frameworks, *J. Combin. Theory Ser. B*, 103:689–705, 2013.
- [8] R. Connelly, Combining globally rigid frameworks, *Proc. Steklov Inst. Math.*, 274:191–198, 2011.
- [9] R. Connelly and W. Whiteley, Global rigidity: The effect of coning. *Discrete Comput. Geom.* 43:717–735, 2010.
- [10] A. Frank and L. Szegő, Constructive characterizations for packing and covering with trees, *Discrete Appl. Math.* 131:347–371, 2003.
- [11] S. Frank and J. Jiang, New classes of counterexamples to Hendrickson’s global rigidity conjecture, *Discrete Comput. Geom.*, 45:574–591, 2011.
- [12] H. Gluck. Almost all simply connected closed surfaces are rigid. In *Geometric topology*, volume 438 of *Lecture Notes in Mathematics*, pages 225–240. Springer, 1975.
- [13] S. Gortler, A. Healy, and D. Thurston, Characterizing generic global rigidity, *American J. Math.*, 132:897–939, 2010.

- [14] S. Gortler and D. Thurston, Generic global rigidity in complex and pseudo-euclidean spaces, *Rigidity and Symmetry, Fields Institute Communications*, 70:131–154, 2014.
- [15] B. Hendrickson, Conditions for unique graph realizations, *SIAM J. Comput.*, 12:65–84, 1992.
- [16] B. Jackson and T. Jordán, Connected rigidity matroids and unique realizations of graphs. *J. Combin. Theory Ser. B*, 94:1–29, 2005.
- [17] B. Jackson, T. Jordán and Z. Szabadka, Globally linked pairs of vertices in equivalent realizations of graphs, *Discrete Comput. Geom.*, 35:493–512, 2006.
- [18] B. Jackson, T. Jordán and Z. Szabadka, Globally linked pairs of vertices in rigid frameworks, *Rigidity and Symmetry, Fields Institute Communications*, 70:177–203, 2014.
- [19] B. Jackson, T. Jordán and S. Tanigawa, Combinatorial conditions for the unique completability of low rank matrices, EGRES Technical Report, TR2014-01, 2014.
- [20] B. Jackson and J. C. Owen, The number of equivalent realisations of a rigid graph, arXiv:1204.1228, 2012.
- [21] T. Jordán, Combinatorial rigidity: graphs and matroids in the theory of rigid frameworks, manuscript.
- [22] T. Jordán, C. Király, S. Tanigawa, Generic global rigidity of body-hinge frameworks, EGRES Technical Reports, TR2014-06, July 2014.
- [23] V. Kaszanitzky and C. Király, On minimally k -rigid graphs. EGRES Technical Report, TR2012-21, 2012.
- [24] N. Katoh and S. Tanigawa, A proof of the molecular conjecture. *Discrete Comput. Geom.*, 45:647–700, 2011.
- [25] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Eng. Math.*, 4(4):331–340, 1970.
- [26] B. Servatius. Birigidity in the plane. *SIAM J. Discrete Math.* 8:582-589, 1989.
- [27] Z. Szabadka. Globally rigid frameworks and rigid tensegrity graphs in the plane. PhD. Thesis, Department of Operations Research, Eötvös Loránd University, 2010.
- [28] S. Tanigawa. Generic rigidity matroids with Dilworth truncations. *SIAM J. Discrete Math.*, 26:1412–1439, 2012.
- [29] S. Tanigawa. Sufficient conditions for the global rigidity of graphs. arXiv:1403.3742, 2014.
- [30] T. Tay. Rigidity of multi-graphs. I: Linking rigid bodies in n -space. *J. Combin. Theory Ser. B*, 36(1):95–112, 1984.
- [31] T. Tay. Linking $(n - 2)$ -dimensional panels in n -space II: $(n - 2, 2)$ -frameworks and body and hinge structures. *Graphs and Combinatorics*, 5(1):245–273, 1989.

- [32] T. Tay. Linking $(n - 2)$ -dimensional panels in n -space I: $(k - 1, k)$ -graphs and $(k - 1, k)$ -frames. *Graphs and Combinatorics*, 7(3):289–304, 1991.
- [33] W. Whiteley. The union of matroids and the rigidity of frameworks. *SIAM J. Discrete Math.*, 1(2):237–255, 1988.
- [34] W. Whiteley. Some matroids from discrete applied geometry. *Contemporary Mathematics*, 197:171–312, 1996.